MIFP-04-02 MCTP-04-09
hep-th/0402223
February 2004

Supergravities with Minkowski × Sphere Vacua

J. Kerimo [†1], James T. Liu [‡2], H. Lü [†1] and C.N. Pope [†1]

[†] *George P. and Cynthia W. Mitchell Institute for Fundamental Physics,*
Texas A& M University, College Station, TX 77843-4242, USA

[‡] *Michigan Center for Theoretical Physics*
University of Michigan, Ann Arbor, MI 48109-1120

ABSTRACT

Recently the authors have introduced a new gauged supergravity theory with a positive definite potential in $D = 6$, obtained through a generalised Kaluza-Klein reduction from $D = 7$. Of particular interest is the fact that this theory admits certain Minkowski×Sphere vacua. In this paper we extend the previous results by constructing gauged supergravities with positive definitive potentials in diverse dimensions, together with their vacuum solutions. In addition, we prove the supersymmetry of the generalised reduction ansatz. We obtain a supersymmetric solution with no form-field fluxes in the new gauged theory in $D = 9$. This solution may be lifted to $D = 10$, where it acquires an interpretation as a time-dependent supersymmetric cosmological solution supported purely by the dilaton. A further uplift to $D = 11$ yields a solution describing a pp-wave.

[1] Research supported in part by DOE grant DE-FG03-95ER40917.

[2] Research supported in part by DOE grant DE-FG02-95ER40899.

Contents

1 Introduction

Recent interest in both de Sitter and anti-de Sitter vacua has led to a renewed study of gauged supergravities, where the gauging of some R-symmetry naturally leads to a non-trivial potential. Well-known examples include the gauged supergravities in four, five and seven dimensions that admit maximally supersymmetric anti-de Sitter vacua. In addition, there are also gauged supergravities with run-away potentials. Although such theories do not admit maximally supersymmetric vacua, they typically allow domain-wall solutions where scalar gradient energy is balanced against the scalar potential. What has not been achieved, however, is the construction of conventional gauged supergravities admitting de Sitter vacua. Of course this is not particularly surprising, since de Sitter spacetime is incompatible with conventional supersymmetry.

Supergravities with positive-definite (albeit run-away) potentials do nevertheless exist. A particularly interesting example is the Salam-Sezgin model, which is a gauged $\mathcal{N} = (1, 0)$ supergravity in $D = 6$ coupled to a tensor and an abelian vector multiplet [1]. This model has a $(\text{Minkowski})_4 \times S^2$ vacuum, in which the vector has a non-trivial flux on the 2-sphere. This monopole flux, combined with the single-exponential potential $V \sim \exp(-\varphi/\sqrt{2})$, is responsible for a "self-tuning" of the vacuum, in which the positive energy density is confined to the 2-sphere, thereby ensuring a vanishing 4-dimensional cosmological constant and correspondingly a $(\text{Minkowski})_4$ vacuum. The self-tuning feature of this model has attracted much attention, especially as a means of protecting the cosmological constant from large corrections even after supersymmetry breaking [2, 3]. It was shown in [4] that the Salam-Sezgin theory arises from a consistent reduction of ten-dimensional supergravity on a circle times a hyperbolic 3-space. It was also shown, in [5], that the Salam-Sezgin model can be consistently reduced on S^2 to give rise to $\mathcal{N} = 1$, $D = 4$ supergravity coupled to an $SU(2)$ vector multiplet and a scalar multiplet.

The interesting features of the Salam-Sezgin model have led us to search for other possible supergravity theories with positive-definite potentials. This search was guided by the realization of [6] that a generalised Kaluza-Klein reduction which gauges a combination of a homogeneous global scaling symmetry together with a

Cremmer-Julia type global symmetry yields a consistent reduction with just such a positive-definite potential. In particular, this generalised reduction was used to construct a variant $\mathcal{N} = (1,1)$ supergravity in $D = 6$ admitting both $(\text{Minkowski})_4 \times S^2$ and $(\text{Minkowski})_3 \times S^3$ vacua [7, 8]. This construction is based on the generalised reduction of minimal $D = 7$ supergravity, where a would-be vector multiplet may be truncated out by a judicious choice of the gauging parameters. In this manner, the reduction takes one from a pure $(d+1)$-dimensional supergravity without a potential to a pure d-dimensional supergravity with a (positive-definite) single-exponential potential. Generalised Kaluza-Klein reduction *via* the gauging of the Cremmer-Julia global symmetries were considered in [9, 10, 11, 12]

Although the work of [7, 8] focused on the reduction from seven to six dimensions, the generalised Kaluza-Klein procedure may be carried out in arbitrary dimensions. In general, the various supergravities in diverse dimensions are quite distinct (especially in their fermionic sectors). However it is noteworthy that the bosonic sector of the half-maximal (16 supercharge) supergravities in $D \le 10$ is universal, with field content

$$(g_{\mu\nu}, B_{\mu\nu}, \phi, A^a_\mu) \tag{1.1}$$

($a = 1, 2, \ldots, 10 - D$). This is of course the bosonic content of the heterotic string (or the NS-NS sector of the Type-II string) compactified on a $(10 - D)$-dimensional torus, with vector multiplets truncated out. Owing to this universality of the field content, we may perform a generalised Kaluza-Klein reduction on the half-maximal supergravities in arbitrary dimensions, and in this manner obtain the full class of (16 supercharge) variant supergravities generalising the results of [7, 8].

The resulting d-dimensional variant supergravities admit both $(\text{Minkowski})_{d-3} \times S^3$ and also, in certain cases, $(\text{Minkowski})_{d-2} \times S^2$, vacua. Furthermore, we are able to construct a new time-dependent supersymmetric solution (or "cosmological solution") in $D = 9$ with no form-field fluxes. This solution lifts to a purely dilaton driven cosmology in $D = 10$, and a pp-wave in $D = 11$.

2 Generalised reduction

We begin with the generalised Kaluza-Klein reduction of the bosonic sector of half-maximal supergravities in arbitrary dimensions $D \le 10$. In this section, all fields and

their equations of motion pertain to the Einstein frame. The string-frame picture will be examined in section 3.

As indicated above, the bosonic field content of pure supergravity with 16 supercharges consists of the graviton $\hat{g}_{\mu\nu}$, antisymmetric tensor $\hat{B}_{\mu\nu}$ and dilaton $\hat{\phi}$, along with $(10-D)$ 1-form potentials $\hat{A}^a_\mu$. The Lagrangian for the bosonic sector can be written as

$$\hat{\mathcal{L}} = \hat{R}\hat{*}\mathbb{1} - \tfrac12 \hat{*}d\hat{\phi}\wedge d\hat{\phi} - \tfrac12 e^{\hat{a}\hat{\phi}}\hat{*}\hat{H}_{(3)}\wedge \hat{H}_{(3)} - \tfrac12 e^{\frac12\hat{a}\hat{\phi}}\hat{*}\hat{F}^a_{(2)}\wedge \hat{F}^a_{(2)}\,, \tag{2.1}$$

where $\hat{F}^a_{(2)} = d\hat{A}^a_{(1)}$, $\hat{H}_{(3)} = d\hat{B}_{(2)} - \frac12 \hat{F}^a_{(2)}\wedge \hat{A}^a_{(1)}$, and $a = 1, 2, \ldots, (10-D)$. The constant $\hat{a}$ is given by

$$\hat{a}^2 = \frac{8}{D-2}\,. \tag{2.2}$$

The equations of motion following from (2.1) are

$$\begin{aligned}
\hat{R}_{MN} &= \tfrac12 \partial_M\hat{\phi}\,\partial_N\hat{\phi} + \tfrac14 e^{\hat{a}\hat{\phi}}\Big(\hat{H}_{MPQ}\,\hat{H}_N{}^{PQ} - \frac{2}{3(D-2)}\hat{H}^2_{(3)}\,\hat{g}_{MN}\Big)\\
&\quad + \tfrac12 e^{\frac12\hat{a}\hat{\phi}}\Big(\hat{F}^a_{MP}\,\hat{F}^a_N{}^P - \frac{1}{2(D-2)}(\hat{F}^a_{(2)})^2\,\hat{g}_{MN}\Big)\,,\\
d(e^{\hat{a}\hat{\phi}}\,\hat{*}\hat{H}_{(3)}) &= 0\,,\\
d(e^{\frac12\hat{a}\hat{\phi}}\,\hat{*}\hat{F}^a_{(2)}) &= (-1)^{D+1} e^{\hat{a}\hat{\phi}}\,\hat{*}\hat{H}_{(3)}\wedge \hat{F}^a_{(2)}\,,\\
\hat{\Box}\hat{\phi} &= \frac{\hat{a}}{12}\,e^{\hat{a}\hat{\phi}}\hat{H}^2_{(3)} + \frac{\hat{a}}{8}\,e^{\frac12\hat{a}\hat{\phi}}\,(\hat{F}^a_{(2)})^2\,.
\end{aligned} \tag{2.3}$$

The key observation behind the generalised reduction of ref. [6] is that the equations of motion are invariant under the two global symmetries

$$\begin{aligned}
\hat{\phi} \to \hat{\phi} + \frac{1}{\hat{a}}\,\lambda_1\,,&\qquad d\hat{s}^2 \to e^{2\lambda_2}\,d\hat{s}^2\,,\\
\hat{B}_{(2)} \to e^{-2\lambda_1+2\lambda_2}\,\hat{B}_{(2)}\,,&\qquad \hat{A}^a_{(1)} \to e^{-\lambda_1+\lambda_2}\,\hat{A}^a_{(1)}\,.
\end{aligned} \tag{2.4}$$

The constant λ_1 parameterises a global symmetry of the Lagrangian, while the scaling transformation parameterised by the constant λ_2 is a symmetry only at the level of the equations of motion, since the Lagrangian scales homogeneously as $\sqrt{-\hat{g}}(\hat{R}+\cdots) \longrightarrow e^{(D-2)\,\lambda_2}\sqrt{-\hat{g}}\,(\hat{R}+\cdots)$.

Following [6], we now reduce from D dimensions to $d = (D-1)$, while simultaneously gauging the above two global symmetries. The D-dimensional pure supergravity multiplet then reduces to d-dimensional supergravity coupled to a single vector

multiplet. This is achieved by making the generalised reduction ansatz

$$\begin{aligned}
d\hat{s}^2 &= e^{2m_2 z}\left(e^{2\alpha\varphi}\, ds^2 + e^{2\beta\varphi}\,(dz + \mathcal{A}_{(1)})^2\right),\\
\hat{B}_{(2)} &= e^{2(m_2-m_1)z}\left(B_{(2)} + B_{(1)} \wedge dz\right),\\
\hat{A}^a_{(1)} &= e^{(m_2-m_1)z}\left(A^a_{(1)} + \chi^a\, dz\right),\\
\hat{\phi} &= \phi + \frac{4}{\hat{a}}\, m_1 z\,,
\end{aligned}\qquad (2.5)$$

where

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}\,,\qquad \beta = -(d-2)\alpha\,. \qquad (2.6)$$

The standard Kaluza-Klein ansatz for an ungauged S^1 reduction would correspond to setting $m_1 = m_2 = 0$.

In general, for unequal mass parameters m_1 and m_2, the lower-dimensional equations of motion are rather complicated. However, a significant simplification occurs if $m_1 = m_2$. In this case, various exponential factors drop out from (2.5), and one can consistently truncate out the vector multiplet, owing to conspiracies between the fields. In this manner, one can obtain variant gauged supergravities with positive-definite scalar potentials and with half-maximal supersymmetry in $d \le 9$ dimensions.

Before writing out the complete reduction of the bosonic equations of motion, we first collect some intermediate results. The reduction of the potentials in (2.5) yields a corresponding reduction on the field strengths:

$$\begin{aligned}
\hat{H}_{(3)} &= e^{2(m_2-m_1)z}(H_{(3)} + H_{(2)} \wedge (dz + \mathcal{A}_{(1)}))\,,\\
\hat{F}^a_{(2)} &= e^{(m_2-m_1)z}(F^a_{(2)} + L^a_{(1)} \wedge (dz + \mathcal{A}_{(1)}))\,,
\end{aligned}\qquad (2.7)$$

where the lower dimensional fields are defined by

$$\begin{aligned}
H_{(3)} &= dB_{(2)} - \tfrac{1}{2}F^a_{(2)}\wedge A^a_{(1)} - dB_{(1)} \wedge \mathcal{A}_{(1)} - 2(m_2-m_1)B_{(2)} \wedge \mathcal{A}_{(1)} + \tfrac{1}{2}\chi^a F^a_{(2)} \wedge \mathcal{A}_{(1)}\,,\\
G_{(2)} &= dB_{(1)} - \tfrac{1}{2}\chi^a\, F^a_{(2)} + \tfrac{1}{2}L^a_{(1)} \wedge A^a_{(1)} - \tfrac{1}{2}\chi^a L^a_{(1)} \wedge \mathcal{A}_{(1)} + 2(m_2-m_1)B_{(2)}\,,\\
F^a_{(2)} &= dA^a_{(1)} - d\chi^a \wedge \mathcal{A}_{(1)} + (m_2-m_1)A^a_{(1)} \wedge \mathcal{A}_{(1)}\,,\\
L^a_{(1)} &= d\chi^a - (m_2-m_1)A^a_{(1)}\,.
\end{aligned}\qquad (2.8)$$

The Kaluza-Klein potential $\mathcal{A}_{(1)}$ has the standard field strength $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. It is evident at this stage that the vector fields $A^a_{(1)}$ and the tensor field $B_{(2)}$ acquire masses proportional to $|m_2 - m_1|$, in the process eating the axions χ^a and the vector $B_{(1)}$ respectively.

2.1 Untruncated d-dimensional equations

We are now able to write down the full bosonic equations of motion for the variant d-dimensional gauged supergravity. The bosonic field content is

$$(g_{\mu\nu}, B_{\mu\nu}, \varphi, A^a_\mu, \mathcal{A}_\mu) \quad \text{and} \quad (B_\mu, \chi^a, \phi)\,, \tag{2.9}$$

corresponding to half-maximal supergravity coupled to a single vector multiplet. This representation is schematic in the sense that the scalars ϕ and φ as well as the 1-form potentials $B_{(1)}$ and $\mathcal{A}_{(1)}$ must necessarily be taken as appropriate linear combinations in the actual multiplets.

We find that the equations of motion for the form fields are given by

$$\begin{aligned}
\nabla^\sigma(e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}) &= (2m_1+(d-3)\,m_2)\left(e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}\mathcal{A}^\sigma - e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\right),\\
\nabla^\nu(e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}\mathcal{F}^{\nu\sigma}\\
&\quad + (2m_1+(d-3)\,m_2)e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\,\mathcal{A}^\nu\,,\\
\nabla^\nu(e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}F^{a\nu\sigma} + e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}L^{a\nu}\\
&\quad + (m_1+(d-2)\,m_2)\left(e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}\mathcal{A}^\nu - e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu\right),\\
\nabla^\mu(e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu) &= -\tfrac12 e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\,F^{a\mu\nu} + \tfrac12 e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}\mathcal{F}^{\mu\nu}\\
&\quad + (m_1+(d-2)\,m_2)e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu\mathcal{A}^\mu\,,\\
\nabla^\nu(e^{-2(d-1)\alpha\varphi}\mathcal{F}_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}G^{\nu\sigma} - e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}L^{a\nu}\\
&\quad + \frac{4}{\hat a}\,m_1(\partial_\mu\phi - \frac{4}{\hat a}\,m_1\mathcal{A}_\mu) - 2m_2(d-1)\,(\beta\partial_\mu\varphi - m_2\mathcal{A}_\mu)\\
&\quad + m_2(d-1)e^{-2(d-1)\alpha\varphi}\,\mathcal{F}_{\mu\nu}\mathcal{A}^\nu\,.
\end{aligned} \tag{2.10}$$

The two scalar fields, ϕ and φ satisfy similar m_1 and m_2 dependent equations of motion. The scalar coming from the metric satisfies the equation

$$\begin{aligned}
-\beta\Box\varphi &= -\frac{e^{\hat a\phi-4\alpha\varphi}}{6(d-1)}H^2_{(3)} - \frac{e^{\frac12\hat a\phi-2\alpha\varphi}}{4(d-1)}(F^a_{(2)})^2 + \frac{d-3}{4(d-1)}e^{\hat a\phi+2(d-3)\alpha\varphi}\,G^2_{(2)}\\
&\quad + \frac{d-2}{2(d-1)}\,e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}(L^a_{(1)})^2 - \tfrac14 e^{-2(d-1)\alpha\varphi}\mathcal{F}^2_{(2)}\\
&\quad - m_2\beta(d-1)\mathcal{A}^\mu\partial_\mu\varphi - m_2\nabla_\mu\mathcal{A}^\mu + m_2^2(d-1)\mathcal{A}^2_{(1)} + \frac{8}{\hat a^2}\,m_1^2e^{2(d-1)\alpha\varphi}\,,
\end{aligned} \tag{2.11}$$

while the D-dimensional dilaton equation reduces to

$$\Box\phi = \frac{\hat a}{12}\,e^{\hat a\phi-4\alpha\varphi}H^2_{(3)} + \frac{\hat a}{4}\,e^{\hat a\phi+2(d-3)\alpha\varphi}G^2_{(2)} + \frac{\hat a}{8}\,e^{\frac12\hat a\phi-2\alpha\varphi}(F^a_{(2)})^2$$

$$
\begin{aligned}
&+\frac{\hat a}{4}\, e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}(L^a_{\scriptscriptstyle(1)})^2 +\, m_2(d-1)\mathcal{A}^\mu\partial_\mu\phi+\frac{4}{\hat a}\, m_1\nabla_\mu\mathcal{A}^\mu\\
&-\frac{4(d-1)}{\hat a}\, m_1 m_2\, (\mathcal{A}^2_{\scriptscriptstyle(1)}+e^{2(d-1)\alpha\varphi})\,. \qquad (2.12)
\end{aligned}
$$

The d-dimensional Einstein equation takes the form

$$
\begin{aligned}
&R_{\mu\nu}-\tfrac12 R g_{\mu\nu} = \tfrac12(\partial_\mu\varphi\,\partial_\nu\varphi-\tfrac12(\partial\varphi)^2\,g_{\mu\nu})+\tfrac12(\partial_\mu\phi\,\partial_\nu\phi-\tfrac12(\partial\phi)^2\,g_{\mu\nu})\\
&+\tfrac12 e^{-2(d-1)\alpha\varphi}\,(\mathcal{F}_{\mu\sigma}\mathcal{F}_\nu{}^\sigma-\tfrac14 g_{\mu\nu}\mathcal{F}^2_{\scriptscriptstyle(2)})+\tfrac14 e^{\hat a\phi-4\alpha\varphi}(H_{\mu\rho\sigma}\,H_\nu{}^{\rho\sigma}-\tfrac16 g_{\mu\nu}H^2_{\scriptscriptstyle(3)})\\
&+\tfrac12 e^{\frac12\hat a\phi-2\alpha\varphi}(F^a_{\mu\sigma}\,F^{a\,\sigma}_\nu-\tfrac14 g_{\mu\nu}(F^a_{\scriptscriptstyle(2)})^2)+\tfrac12 e^{\hat a\phi+2(d-3)\alpha\varphi}(G_{\mu\sigma}G_\nu{}^\sigma-\tfrac14 g_{\mu\nu}G^2_{\scriptscriptstyle(2)})\\
&+\tfrac12 e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}\,(L^a_\mu L^a_\nu-\tfrac12 g_{\mu\nu}\,(L^a_{\scriptscriptstyle(1)})^2)\\
&-\alpha m_2(d-1)(\mathcal{A}^\sigma\partial_\sigma\varphi\, g_{\mu\nu}-\mathcal{A}_\mu\partial_\nu\varphi-\mathcal{A}_\nu\partial_\mu\varphi)\\
&+\frac{2}{\hat a}\, m_1(\mathcal{A}^\sigma\partial_\sigma\phi\, g_{\mu\nu}-\mathcal{A}_\mu\partial_\nu\phi-\mathcal{A}_\nu\partial_\mu\phi)+\Big(\frac{8}{\hat a^2}\, m_1^2-(d-1)m_2^2\Big)\mathcal{A}_\mu\mathcal{A}_\nu\\
&-\tfrac12 m_2(d-1)(\nabla_\mu\mathcal{A}_\nu+\nabla_\nu\mathcal{A}_\mu-2\nabla_\sigma\mathcal{A}^\sigma\, g_{\mu\nu})\\
&-\Big(\frac{4m_1^2}{\hat a^2}+\tfrac12 m_2^2(d-1)(d-2)\Big)\,(\mathcal{A}^2_{\scriptscriptstyle(1)}+e^{2(d-1)\alpha\varphi})g_{\mu\nu}\,. \qquad (2.13)
\end{aligned}
$$

Note that the last term is associated with a positive-definite scalar potential.

2.2 Truncated d-dimensional equations

The scalars ϕ and φ may be disentangled between the supergravity and vector multiplets of (2.9) by performing a rotation to ϕ_1 (supergravity) and ϕ_2 (vector) given by

$$
\hat a\phi-4\alpha\varphi=a\phi_1\,,\qquad 4\alpha\phi+\hat a\varphi=a\phi_2\,, \qquad (2.14)
$$

where $a=\sqrt{8/(D-3)}$. When $m_1=m_2$, the vector multiplet may be further truncated away. This is done by setting

$$
B_{\scriptscriptstyle(1)}=\mathcal{A}_{\scriptscriptstyle(1)}\equiv\tfrac{1}{\sqrt2}A_{\scriptscriptstyle(1)}\,,\qquad \phi_2=0\,,\qquad L^a_{\scriptscriptstyle(1)}=0\,. \qquad (2.15)
$$

The equations of motion for the pure supergravity fields are then given by

$$
\begin{aligned}
&\nabla^\rho\Big(e^{a\phi}H_{\mu\nu\rho}\Big)=\frac{d-1}{\sqrt2}\, m\Big(e^{a\phi}\, H_{\mu\nu\rho}\, A^\rho-e^{\frac12 a\phi}F_{\mu\nu}\Big)\,,\\
&\nabla^\nu\Big(e^{\frac12 a\phi}F_{\mu\nu}\Big)=\tfrac12 e^{a\phi}H_{\mu\nu\rho}F^{\nu\rho}+\frac{d-1}{\sqrt2}\, me^{\frac12 a\phi}F_{\mu\nu}\, A^\nu\,,\\
&\nabla^\nu\Big(e^{\frac12 a\phi}F^a_{\mu\nu}\Big)=\tfrac12 e^{a\phi}H_{\mu\nu\rho}F^{a\,\nu\rho}+\frac{d-1}{\sqrt2}\, me^{\frac12 a\phi}F^a_{\mu\nu}\, A^\nu\,,
\end{aligned}
$$

$$
\begin{aligned}
\Box\phi &= \frac{e^{a\phi}}{3\sqrt{2(d-2)}} H_{(3)}^2 + \frac{e^{\frac{1}{2}a\phi}}{2\sqrt{2(d-2)}} (F_{(2)}^2 + (F_{(2)}^a)^2) + \frac{d-1}{\sqrt{2}}\, mA^\mu\, \partial_\mu\phi \\
&\quad + \frac{d-1}{\sqrt{d-2}}\, m\nabla_\mu\, A^\mu - \frac{\sqrt{2}\,(d-1)^2}{\sqrt{d-2}}\, m^2 (\tfrac{1}{2} A_{(1)}^2 + e^{-\frac{1}{2}a\phi})\,, \\
R_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{4} e^{a\phi}(H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - \frac{2}{3(d-2)} H_{(3)}^2 g_{\mu\nu}) \\
&+ \tfrac{1}{2} e^{\frac{1}{2}a\phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(d-2)} F_{(2)}^2 g_{\mu\nu}) + \tfrac{1}{2} e^{\frac{1}{2}a\phi}(F_{\mu\rho}^a F_\nu^{a\,\rho} - \frac{1}{2(d-2)} (F_{(2)}^a)^2 g_{\mu\nu}) \\
&- \frac{m(d-1)}{2\sqrt{d-2}}(A_\mu\partial_\nu\phi + A_\nu\partial_\mu\phi) - \frac{m(d-1)}{2\sqrt{2}}(\nabla_\mu A_\nu + \nabla_\nu A_\mu + \frac{2}{d-2}\nabla_\rho A^\rho g_{\mu\nu}) \\
&+ \frac{m^2(d-1)^2}{2(d-2)}(A_{(1)}^2 + 2e^{-\frac{1}{2}a\phi}) g_{\mu\nu}\,, \qquad (2.16)
\end{aligned}
$$

where we have rewritten ϕ_1 as ϕ. It may be seen that this set of equations cannot be obtained from a Lagrangian in terms of the physical fields. This is not altogether surprising, since they were derived in a generalised reduction that gauged a symmetry of the equations of motion which was not a symmetry of the Lagrangian.

By examining the linearised equations of motion, it can be seen that $A_{(1)}$ is a massless gauge potential. This gauge field can in fact be consistently set to zero. In this case, the remaining equations of motion can then be obtained from the Lagrangian

$$
e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12} e^{a\phi} H_{(3)}^2 - \tfrac{1}{4} e^{\frac{1}{2}a\phi} (F_{(2)}^a)^2 - (d-1)^2\, m^2 e^{-\frac{1}{2}a\phi}\,, \qquad (2.17)
$$

where $e = \sqrt{-g}$. Thus we see once again that the scalar potential is positive definite.

3 String frame and σ-model action

For many purposes it is advantageous to perform the Weyl rescaling of the metric that transforms from the Einstein frame that we used in the previous section to the string frame. One reason is because the half-maximal supergravities that we are considering have a direct relation to the heterotic string, or the NS-NS sector of the Type-II string. Another reason is that many of the formulae become considerably simpler when expressed in the string frame. We shall consider only the case $m_1 = m_2 = m$.

Consistent string propagation demands world-sheet conformal invariance, and hence the vanishing of the beta functions for the background spacetime fields. In this manner one obtains supergravity equations of motion which arise naturally in

the string frame. The corresponding equations may be derived from the string-frame Lagrangian

$$\hat{e}^{-1}\hat{\mathcal{L}} = e^{-2\hat{\Phi}}(\hat{R} + 4(\partial\hat{\Phi})^2 - \tfrac{1}{12}\hat{H}_{(3)}^2 - \tfrac{1}{4}(\hat{F}^a_{(2)})^2)\,, \tag{3.1}$$

taken here to have been compactified on a $(10-D)$-dimensional torus (with the additional truncation of $(10-D)$ vector multiplets). It is to be understood that all fields in this section are labelled with a suppressed tilde ($\tilde{g}_{\mu\nu}$, $\widetilde{H}_{(3)}$, *etc.*) unless otherwise indicated, to distinguish them from the Einstein frame fields. The complete transformation between the two frames in dimensions $D \le 10$ is given in appendix C.

The equations of motion following from the Lagrangian (3.1) are

$$\begin{aligned}
\hat{R}_{MN} &= -2\widehat{\nabla}_M\widehat{\nabla}_N\hat{\Phi} + \tfrac{1}{4}\hat{H}_{MPQ}\hat{H}_N{}^{PQ} + \tfrac{1}{2}\hat{F}^a_{MP}\hat{F}^{a\ P}_N\,,\\
d(e^{-2\hat{\Phi}}\hat{*}\hat{H}_{(3)}) &= 0\,,\\
d(e^{-2\hat{\Phi}}\hat{*}\hat{F}^a_{(2)}) &= (-1)^{D+1}\,e^{-2\hat{\Phi}}\hat{*}\hat{H}_{(3)}\wedge\hat{F}^a_{(2)}\,,\\
\widehat{\Box}\hat{\Phi} &= 2(\partial\hat{\Phi})^2 - \tfrac{1}{12}\hat{H}_{(3)}^2 - \tfrac{1}{8}(\hat{F}^a_{(2)})^2\,.
\end{aligned} \tag{3.2}$$

By tracing the Einstein equation and substituting in the dilaton equation, we may obtain an expression for the Ricci scalar:

$$\hat{R} = -4(\partial\,\hat{\Phi})^2 + \tfrac{5}{12}\hat{H}_{(3)}^2 + \tfrac{3}{4}(\hat{F}^a_{(2)})^2\,. \tag{3.3}$$

In D dimensions, the Einstein-frame and the string-frame metrics are related by

$$d\hat{s}^2_{\rm Ein} = e^{\frac{1}{2}\hat{a}\hat{\phi}}\,d\hat{s}^2_{\rm str} = e^{-\frac{1}{2}\hat{a}^2\hat{\Phi}}\,d\hat{s}^2_{\rm str}\,, \tag{3.4}$$

where we have defined $\hat{\Phi} = -\hat{\phi}/\hat{a}$ and $\hat{\phi}$ is the Einstein-frame dilaton field. For the case where $m_1 = m_2$, the reduction ansatz (2.5) converted to the string frame is rather simple, namely

$$\begin{aligned}
d\hat{s}^2_{\rm str} &= ds^2_{\rm str} + e^{-\sqrt{2}\varphi}(dz + \mathcal{A}_{(1)})^2\,,\\
\hat{B}_{(2)} &= B_{(2)} + B_{(1)}\wedge dz\,,\\
\hat{\Phi} &= \Phi - \tfrac{1}{\sqrt{8}}\,\varphi - \tfrac{1}{2}(d-1)\,mz\,.
\end{aligned} \tag{3.5}$$

In other words, the reduction is exactly the same as a standard Kaluza-Klein reduction, except for a linear z-dependence in the dilaton $\hat{\Phi}$.

It follows that the σ-model action for this generalised circle reduction is given by

$$
\begin{aligned}
I &= \frac{1}{4\pi\,\alpha'}\int d\sigma\, d\tau\Big[\sqrt{\gamma}\,\gamma^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,\hat g_{\mu\nu} + \epsilon^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,\hat B_{\mu\nu} \\
&\qquad + \alpha' \hat R\,(\Phi - \tfrac12 (D-2)\, m z)\Big]\,,
\end{aligned}
$$

where Φ, $\hat g_{\mu\nu}$ and $\hat B_{\mu\nu}$ are independent of z, and X^0 (the circle coordinate) is given by $X^0 = z$. However, the z dependence of the string action implies that T-duality is now broken. This can also be seen from the low-energy effective action obtained in the previous section, where the Kaluza-Klein vector $\mathcal{A}_{(1)}$ and the winding vector $B_{(1)}$ are clearly not on a parallel footing.

3.1 Untruncated d-dimensional string-frame equations

We give here the complete set of bosonic equations of motion for the untruncated system, expressed in the string frame. It will be seen that these are considerably simpler than the previous expressions that were obtained in the Einsten frame.

For the form fields in the string frame we find

$$
\begin{aligned}
\nabla^\rho(e^{-2\Phi} H_{\mu\nu\rho}) &= m(d-1)\Big(e^{-2\Phi} H_{\mu\nu\sigma}\mathcal{A}^\sigma - e^{-2\Phi+\sqrt2\varphi} G_{\mu\nu}\Big)\,, \\
\nabla^\nu(e^{-2\Phi+\sqrt2\varphi} G_{\mu\nu}) &= \tfrac12 e^{-2\Phi} H_{\mu\nu\sigma}\mathcal{F}^{\nu\sigma} + m(d-1) e^{-2\Phi+\sqrt2\varphi} G_{\mu\nu}\mathcal{A}^\nu\,, \\
\nabla^\nu(e^{-2\Phi} F^a_{\mu\nu}) &= \tfrac12 e^{-2\Phi} H_{\mu\nu\sigma} F^{a\,\nu\sigma} + e^{-2\Phi+\sqrt2\varphi} G_{\mu\nu} L^{a\,\nu} \\
&\quad + m(d-1)\Big(e^{-2\Phi} F^a_{\mu\nu}\mathcal{A}^\nu - e^{-2\Phi+\sqrt2\varphi} L^a_\mu\Big)\,, \\
\nabla^\mu(e^{-2\Phi+\sqrt2\varphi} L^a_\mu) &= \tfrac12 e^{-2\Phi} F^a_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac12 e^{-2\Phi+\sqrt2\varphi} G_{\mu\nu} F^{a\,\mu\nu} \\
&\quad + m(d-1) e^{-2\Phi+\sqrt2\varphi} L^a_\mu \mathcal{A}^\mu\,, \\
\nabla^\nu(e^{-\frac{3}{\sqrt2}\varphi}\mathcal{F}_{\mu\nu}) &= e^{-\frac{1}{\sqrt2}\varphi}(\tfrac12 H_{\mu\nu\sigma} G^{\nu\sigma} - F^a_{\mu\nu} L^{a\nu}) + 2 e^{-\frac{3}{\sqrt2}\varphi}(\partial_\nu\Phi - \tfrac{1}{\sqrt8}\partial_\nu\varphi)\mathcal{F}_\mu{}^\nu \\
&\quad + m(d-1)(\sqrt2\, e^{-\frac{1}{\sqrt2}\varphi}\partial_\mu\varphi + e^{-\frac{3}{\sqrt2}\varphi}\mathcal{A}^\nu\mathcal{F}_{\mu\nu})\,.
\end{aligned}
\tag{3.6}
$$

For the scalar fields, we find

$$
\begin{aligned}
\Box\varphi &= \tfrac{1}{2\sqrt2}(e^{\sqrt2\varphi} G^2_{(2)} - e^{-\sqrt2\varphi}\mathcal{F}^2_{(2)}) + \tfrac{1}{\sqrt2} e^{\sqrt2\varphi}(L^a_{(1)})^2 + 2\partial_\mu\varphi\partial^\mu\Phi + m(d-1)\mathcal{A}^\mu\partial_\mu\varphi\,, \\
\Box\Phi &= -\tfrac{1}{12} H^2_{(3)} - \tfrac18 (F^a_{(2)})^2 - \tfrac18(e^{\sqrt2\varphi} G^2_{(2)} + e^{-\sqrt2\varphi}\mathcal{F}^2_{(2)}) + 2(\partial\Phi)^2 \\
&\quad + 2m(d-1)\mathcal{A}^\mu\,\partial_\mu\Phi - \tfrac12 m(d-1)\nabla_\mu\mathcal{A}^\mu + \tfrac12 m^2(d-1)^2(\mathcal{A}^2_{(1)} + e^{\sqrt2\varphi})\,.
\end{aligned}
\tag{3.7}
$$

The Einstein equations in the string frame are given by

$$
R_{\mu\nu} = \tfrac12\partial_\mu\varphi\partial_\nu\varphi - 2\nabla_\mu\partial_\nu\Phi + \tfrac14 H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + \tfrac12 e^{\sqrt2\varphi} G_{\mu\rho} G_\nu{}^\rho + \tfrac12 e^{-\sqrt2\varphi}\mathcal{F}_{\mu\rho}\mathcal{F}_\nu{}^\rho
$$

$$+\tfrac{1}{2}F^a_{\mu\rho}\,F^{a\,\rho}_{\nu}+\tfrac{1}{2}e^{\sqrt{2}\varphi}L^a_\mu L^a_\nu-\tfrac{1}{2}m(d-1)(\nabla_\mu\mathcal{A}_\nu+\nabla_\nu\mathcal{A}_\mu)\,. \tag{3.8}$$

3.2 Truncated d-dimensional string-frame equations

In the string frame, we may again truncate out the vector multiplet by setting $\varphi=0$, $L^a_{(1)}=0$ and $\mathcal{A}_{(1)}=B_{(1)}\equiv A_{(1)}/\sqrt{2}$. The equations of motion for the bosonic fields of the pure supergravity multiplet now become

$$\begin{aligned}
\nabla^\sigma H_{\mu\nu\sigma} &= 2H_{\mu\nu\sigma}M^\sigma-\tfrac{1}{\sqrt{2}}\,m(d-1)F_{\mu\nu}\,,\\
\nabla^\nu F_{\mu\nu} &= \tfrac{1}{2}H_{\mu\nu\sigma}F^{\nu\sigma}+2F_{\mu\nu}M^\nu\,,\\
\nabla^\nu F^a_{\mu\nu} &= \tfrac{1}{2}H_{\mu\nu\sigma}F^{a\,\nu\sigma}+2F^a_{\mu\nu}M^\nu\,,\\
\nabla^\mu M_\mu &= 2M^2_{(1)}-\tfrac{1}{12}H^2_{(3)}-\tfrac{1}{8}(F^2_{(2)}+(F^a_{(2)})^2)+\tfrac{1}{2}m^2(d-1)^2\,,\\
R_{\mu\nu} &= -\nabla_\mu M_\nu-\nabla_\nu M_\mu+\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}+\tfrac{1}{2}(F_{\mu\rho}\,F_\nu{}^\rho+F^a_{\mu\rho}\,F^{a\,\rho}_\nu)\,,
\end{aligned} \tag{3.9}$$

where we have introduced the field

$$M_{(1)}=d\Phi+\frac{m(d-1)}{2\sqrt{2}}A_{(1)}\,. \tag{3.10}$$

It is evident that the massive field $M_{(1)}$ arises because the dilaton Φ is eaten by the gauge field $A_{(1)}$.

As in the Einstein frame, these equations cannot be obtained from a Lagrangian. However, if we set $A_{(1)}$ to zero, the equations of motion for the remaining fields can be obtained from a Lagrangian, given by

$$e^{-1}\mathcal{L}=e^{-2\Phi}\Big(R+4(\partial\Phi)^2-\tfrac{1}{12}H^2_{(3)}-\tfrac{1}{4}(F^a_{(2)})^2-(d-1)^2m^2\Big)\,. \tag{3.11}$$

Although this truncation is consistent within the bosonic theory, it cannot be consistent with the full supergravity, as it would be incompatible with the structure of the supermultiplets. Nevertheless, we see from (3.11) that in the string frame the scalar potential becomes a pure positive cosmological constant.

4 Supersymmetry

With the derivation of the bosonic equations of motion both in the Einstein frame and the string frame completed, we now turn to a consideration of the supersymmetry

transformation rules for these generalised reductions. We shall present the results for two cases in this section. The first is the variant ten-dimensional massive gauged supergravity obtained in [6] by performing a generalised reduction of eleven-dimensional supergravity.[1] The reduction in this case involves just the global scaling symmetry of the $D = 11$ equations of motion. Then, we shall consider the nine-dimensional massive gauged theory obtained from massless $\mathcal{N} = 1$, $D = 10$ supergravity, using the generalised reduction involving the two global symmetries that we discussed in section 2. Analogous results for the six-dimensional gauged theory were obtained in detail in [8].

4.1 Massive type IIA supergravity from $D = 11$

The supersymmetry transformations in $D = 11$ are

$$\delta\hat{e}_M{}^A = \hat{\bar{\epsilon}}\hat{\gamma}^A\hat{\psi}_M\,, \qquad \delta\hat{A}_{MNP} = 3\hat{\bar{\epsilon}}\hat{\gamma}_{[MN}\hat{\psi}_{P]}\,,$$
$$\delta\hat{\psi}_M = \widehat{\nabla}_M\hat{\epsilon} - \tfrac{1}{288}\hat{F}_{NPQR}(\hat{\gamma}_M{}^{NPQR} - 8\hat{\gamma}^{PQR}\delta^N_M)\,\hat{\epsilon}\,, \tag{4.1}$$

where in our conventions

$$\{\hat{\gamma}_A, \hat{\gamma}_B\} = 2\hat{\eta}_{AB} \tag{4.2}$$

and the metric signature is $(- + + \cdots +)$. The equations of motion of the eleven-dimensional theory are invariant under a scaling symmetry, which was used in [6] in a generalised reduction to obtain the bosonic sector of a massive ten-dimensional supergravity. Here, we extend that discussion to include the fermionic sector. This variant maximal supersymmetric $D = 10$ massive theory [13, 6] has also been considered in [12]. The corresponding ansatz for the generalised circle reduction of the fermions is

$$\begin{aligned}
\hat{\epsilon} &= e^{\frac12 m_2 z} e^{\frac{1}{24}\varphi}\,\epsilon\,,\\
\hat{\psi}_{11} &= \tfrac{2\sqrt2}{3} e^{-\frac12 m_2 z} e^{-\frac{1}{24}\varphi}\hat{\gamma}_{11}\lambda\,,\\
\hat{\psi}_a &= e^{-\frac12 m_2 z} e^{-\frac{1}{24}\varphi}(\psi_a - \tfrac{\sqrt2}{12}\gamma_a\lambda)\,.
\end{aligned} \tag{4.3}$$

[1]Note that this massive type IIA supergravity [13, 6] is not the same as the massive IIA theory obtained by Romans [14].

Performing the reduction of the fermionic transformation rules, we obtain

$$\begin{aligned}
\delta\lambda &= -\tfrac{1}{2\sqrt{2}}\gamma^\mu\epsilon\,\partial_\mu\varphi - \tfrac{1}{192\sqrt{2}}e^{-\frac{1}{4}\varphi}F_{\mu\nu\sigma\rho}\gamma^{\mu\nu\sigma\rho}\epsilon + \tfrac{1}{24\sqrt{2}}e^{\frac{1}{2}\varphi}F_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\hat\gamma_{11}\epsilon \\
&\quad - \tfrac{3}{16\sqrt{2}}e^{-\frac{3}{4}\varphi}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\hat\gamma_{11}\epsilon - \tfrac{3}{4\sqrt{2}}m_2(\mathcal{A}_\mu\gamma^\mu - e^{\frac{3}{4}\varphi}\hat\gamma_{11})\epsilon\,, \\
\delta\psi_\mu &= \nabla_\mu\epsilon - \tfrac{1}{256}e^{-\frac{1}{4}\varphi}F_{\nu\alpha\sigma\rho}\Big(\gamma_\mu{}^{\nu\alpha\sigma\rho} - \tfrac{20}{3}\delta_\mu^\nu\gamma^{\alpha\sigma\rho}\Big)\epsilon - \tfrac{1}{96}e^{\frac{1}{2}\varphi}F_{\nu\sigma\rho}\Big(\gamma_\mu{}^{\nu\sigma\rho} - 9\delta_\mu^\nu\gamma^{\sigma\rho}\Big)\hat\gamma_{11}\epsilon \\
&\quad -\tfrac{1}{64}e^{-\frac{3}{4}\varphi}\mathcal{F}_{\nu\sigma}\Big(\gamma_\mu{}^{\nu\sigma} - 14\delta_\mu^\nu\gamma^\sigma\Big)\hat\gamma_{11}\epsilon - \tfrac{9}{16}m_2(\mathcal{A}_\nu\gamma_\mu\gamma^\nu - e^{\frac{3}{4}\varphi}\gamma_\mu\hat\gamma_{11})\epsilon\,.
\end{aligned} \tag{4.4}$$

The supersymmetry transformation rules for the bosons are

$$\begin{aligned}
&\delta e_\mu{}^a = \bar\epsilon\gamma^a\psi_\mu\,, \qquad \delta\phi = -\sqrt{2}\,\bar\epsilon\,\lambda\,, \\
&\delta\mathcal{A}_\mu = e^{\frac{3}{4}\phi}\bar\epsilon\hat\gamma_{11}(\psi_\mu - \tfrac{3\sqrt{2}}{4}\gamma_\mu\lambda)\,, \\
&\delta A_{\mu\nu} = e^{-\frac{1}{2}\phi}\bar\epsilon\hat\gamma_{11}(2\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{\sqrt{2}}\gamma_{\mu\nu}\lambda)\,, \\
&\delta A_{\mu\nu\rho} = 3e^{\frac{1}{4}\phi}\bar\epsilon(\gamma_{[\mu\nu}\psi_{\rho]} - \tfrac{\sqrt{2}}{12}\gamma_{\mu\nu\rho}\lambda) + 3\mathcal{A}_{[\mu}\delta A_{\nu\rho]}\,.
\end{aligned} \tag{4.5}$$

As was shown in [6] this theory admits a de Sitter vacuum solution, which necessarily breaks all supersymmetry. Note that the ten dimensional field strengths are those defined in [6].

4.2 Reduction of $D = 10, \mathcal{N} = 1$ supersymmetry

Since we have obtained the transformation rules for the type IIA massive gauged supergravity in section 4.1, it is convenient to make use of these here in order to establish our conventions and notation for the transformation rules of the standard massless $\mathcal{N} = 1$ supergravity in ten dimensions. These are obtained by setting the mass parameter $m_2 = 0$ in (4.4), and in addition making the chiral projection that reduces the $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$:

$$\hat\gamma_{11}\epsilon = \epsilon\,, \qquad \hat\gamma_{11}\psi_a = \psi_a \qquad \text{and} \qquad \hat\gamma_{11}\lambda = -\lambda\,. \tag{4.6}$$

The chirality condition is consistent with setting to zero both the 3-form potential and the Kaluza-Klein vector. This yields the ten-dimensional $\mathcal{N} = 1$ supersymmetry transformation rules

$$\begin{aligned}
\delta\hat\lambda &= -\tfrac{1}{2\sqrt{2}}\,\hat\gamma^M\hat\epsilon\,\partial_M\hat\phi + \tfrac{1}{24\sqrt{2}}\,e^{\frac{1}{2}\hat\phi}\,\hat H_{MNP}\hat\gamma^{MNP}\,\hat\epsilon\,, \\
\delta\hat\psi_M &= \widehat\nabla_M\hat\epsilon - \tfrac{1}{96}\,e^{\frac{1}{2}\hat\phi}\hat H_{NPQ}\Big(\hat\gamma_M{}^{NPQ} - 9\,\hat\gamma^{PQ}\delta_M^N\Big)\hat\epsilon\,,
\end{aligned}$$

$$
\begin{aligned}
\delta \hat{e}_M{}^A &= \hat{\bar{\epsilon}}\hat{\gamma}^A\hat{\psi}_M\,, \qquad \delta\hat{\phi} = -\sqrt{2}\,\hat{\bar{\epsilon}}\,\hat{\lambda}\,,\\
\delta \hat{B}_{MN} &= -e^{-\frac{1}{2}\hat{\phi}}\hat{\bar{\epsilon}}\,(2\hat{\gamma}_{[M}\hat{\psi}_{N]} + \tfrac{1}{\sqrt{2}}\hat{\gamma}_{MN}\hat{\lambda})\,.
\end{aligned} \tag{4.7}
$$

We can now use these standard $\mathcal{N}=1$ results in a generalised circle reduction to $d=9$. We shall focus just on the pure supergravity multiplet in $d=9$, by performing a (consistent) truncation of the matter multiplet. The required reduction ansatz is obtained from the arbitrary-dimension ansatz of appendix B by setting $m_1 = m_2 = m$ and $\phi_2 = 0 = \chi$. This gives

$$
\begin{aligned}
\hat{\epsilon} &= e^{\frac{1}{2}mz} e^{-\frac{1}{16\sqrt{14}}\phi_1}\,\tilde{\epsilon}\,,\\
\hat{\lambda} &= \sqrt{\tfrac{7}{8}}\,e^{-\frac{1}{2}mz} e^{\frac{1}{16\sqrt{14}}\phi_1}\,\tilde{\lambda}\,,\\
\hat{\psi}_{10} &= -\tfrac{\sqrt{7}}{8}\,e^{-\frac{1}{2}mz} e^{\frac{1}{16\sqrt{14}}\phi_1}\tilde{\gamma}_{10}\tilde{\lambda}\,,\\
\hat{\psi}_a &= e^{-\frac{1}{2}mz} e^{\frac{1}{16\sqrt{14}}\phi_1}\Big(\tilde{\psi}_a + \tfrac{1}{8\sqrt{7}}\,\tilde{\gamma}_a\tilde{\lambda}\Big)\,,\\
\hat{\phi} &= \tfrac{\sqrt{14}}{4}\,\phi_1 + 4mz\,.
\end{aligned} \tag{4.8}
$$

The tildes signify that the fermions and the Dirac matrices are still ten-dimensional. These can be related to the nine-dimensional quantities as follows:

$$
\begin{aligned}
&\tilde{\gamma}_a = \gamma_a \times \sigma_1\,, \qquad \tilde{\gamma}_{10} = \mathbb{1}\times\sigma_2 \qquad \text{and} \qquad \hat{\gamma}_{11} = \mathbb{1}\times\sigma_3\,,\\
&\tilde{\epsilon} = \epsilon\times\eta\,, \qquad \tilde{\lambda} = \lambda\times\sigma_1\eta \qquad \text{and} \qquad \tilde{\psi}_a = \psi_a\times\eta\,,
\end{aligned} \tag{4.9}
$$

where η is a 2-component constant spinor. The chiral projections (4.6) imply that we must have $\sigma_3\eta = \eta$. In the following subsections, we present the resulting nine-dimensional transformation rules in the Einstein frame and the string frame.

4.2.1 $D = 9$ supersymmetry in the Einstein frame

Reducing the $\mathcal{N}=1$, $D=10$ transformation rules, and setting $G_{(2)} = \mathcal{F}_{(2)} = \frac{1}{\sqrt{2}}\,F_{(2)}$, we obtain the following nine-dimensional supersymmetry transformation rules:

$$
\begin{aligned}
\delta\lambda &= -\tfrac{1}{2\sqrt{2}}\gamma^{\mu}\epsilon\,\partial_\mu\phi + \tfrac{1}{12\sqrt{7}}e^{\sqrt{\frac{2}{7}}\phi}H_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\epsilon + \tfrac{\mathrm{i}}{4\sqrt{14}}e^{\frac{1}{\sqrt{14}}\phi}F_{\mu\nu}\gamma^{\mu\nu}\epsilon\\
&\quad + \tfrac{4}{\sqrt{7}}m\,\Big(\tfrac{1}{\sqrt{2}}\gamma^{\mu}A_\mu - \mathrm{i}e^{-\frac{1}{\sqrt{14}}\phi}\Big)\epsilon\,,\\
\delta\psi_\mu &= \nabla_\mu\epsilon - \tfrac{1}{84}e^{\sqrt{\frac{2}{7}}\phi}H_{\nu\sigma\rho}(\gamma_\mu{}^{\nu\sigma\rho} - \tfrac{15}{2}\delta^{\nu}_{\mu}\gamma^{\sigma\rho})\epsilon - \tfrac{\mathrm{i}}{28\sqrt{2}}e^{\frac{1}{\sqrt{14}}\phi}F_{\nu\sigma}(\gamma_\mu{}^{\nu\sigma} - 12\delta^{\nu}_{\mu}\gamma^{\sigma})\epsilon\\
&\quad - \tfrac{4}{7\sqrt{2}}mA_\nu\gamma_\mu\gamma^{\nu}\epsilon + \tfrac{4\mathrm{i}}{7}me^{-\frac{1}{\sqrt{14}}\phi}\gamma_\mu\epsilon\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon\gamma^a\psi_\mu\,, \qquad \delta\phi = -\sqrt{2}\,\bar\epsilon\,\lambda\,,\\
\delta A_\mu &= \mathrm{i}\sqrt{2}e^{-\frac{1}{\sqrt{14}}\phi}\bar\epsilon(\psi_\mu+\tfrac{1}{\sqrt7}\gamma_\mu\lambda)\,,\\
\delta B_{\mu\nu} &= -e^{-\sqrt{\frac{2}{7}}\phi}\bar\epsilon(2\gamma_{[\mu}\psi_{\nu]}+\tfrac{2}{\sqrt7}\gamma_{\mu\nu}\lambda)-A_{[\mu}\delta A_{\nu]}\,,
\end{aligned}
\tag{4.10}
$$

where we have dropped the "1" subscript on the scalar field. The field strengths are $H_{\mu\nu\rho}=3\partial_{[\mu}B_{\nu\rho]}-\frac{3}{2}A_{[\mu}F_{\nu\rho]}$ and $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$. This theory is an Abelian gauged version of $\mathcal{N}=1, D=9$ supergravity. We shall show that it admits a supersymmetric (Minkowski)$_6\times S^3$ vacuum solution. We shall also obtain a time-dependent supersymmetric cosmological solution in this theory.

4.2.2 $D=9$ supersymmetry in the string frame

The above transformation rules for the fermions are readily expressed in terms of the fields of the string frame, using the formulae given in appendix C. Specialised to nine dimensions, these are

$$
\begin{aligned}
&g_{\mu\nu}=e^{\sqrt{\frac{2}{7}}\phi_1}\tilde g_{\mu\nu}\,, \qquad F_{(2)}=\widetilde F_{(2)}\,, \qquad H_{(3)}=\widetilde H_{(3)}\,, \qquad d\Phi+\sqrt{8}mA_{(1)}=\widetilde M_{(1)}\,,\\
&\phi_1=-\sqrt{\tfrac{8}{7}}\,\Phi\,, \quad \epsilon=e^{\frac{1}{2\sqrt{14}}\phi_1}\tilde\epsilon\,, \quad \lambda=e^{-\frac{1}{2\sqrt{14}}\phi_1}\tilde\lambda\,, \quad \psi_\mu=e^{\frac{1}{2\sqrt{14}}\phi_1}\tilde\psi_\mu\,,
\end{aligned}
\tag{4.11}
$$

The fermionic transformation rules in the string frame then take the form

$$
\begin{aligned}
\delta\tilde\lambda &= \Big(\tfrac{1}{\sqrt7}\widetilde M_\mu\tilde\gamma^\mu+\tfrac{1}{12\sqrt7}\widetilde H_{\mu\nu\sigma}\tilde\gamma^{\mu\nu\sigma}+\tfrac{\mathrm{i}}{4\sqrt{14}}\widetilde F_{\mu\nu}\tilde\gamma^{\mu\nu}-\tfrac{4\mathrm{i}}{\sqrt7}m\Big)\tilde\epsilon\,,\\
\delta\tilde\psi_\mu &= \Big(\widetilde\nabla_\mu-\tfrac{1}{7}\widetilde M_\nu\tilde\gamma_\mu\tilde\gamma^\nu-\tfrac{1}{84}\widetilde H_{\nu\sigma\rho}(\tilde\gamma_\mu{}^{\nu\sigma\rho}-\tfrac{15}{2}\delta_\mu^\nu\tilde\gamma^{\sigma\rho})\\
&\qquad -\tfrac{\mathrm{i}}{28\sqrt2}\widetilde F_{\nu\sigma}(\tilde\gamma_\mu{}^{\nu\sigma}-12\delta_\mu^\nu\tilde\gamma^\sigma)+\tfrac{4\mathrm{i}}{7}m\tilde\gamma_\mu\Big)\tilde\epsilon\,.
\end{aligned}
\tag{4.12}
$$

5 Supersymmetric $M_{d-3}\times S^3$ and $M_{d-2}\times S^2$ vacua

The generalised Kaluza-Klein reduction gives rise to gauged supergravities that admit supersymmetric vacuum solutions of the form Minkowski×Sphere [8]. The nine-dimensional theory admits just a (Minkowski)$_6\times S^3$ vacuum of this kind, supported by the $H_{(3)}$ flux. The theories in lower dimensions admit (Minkowski)$_{d-3}\times S^3$ vacua supported by $H_{(3)}$, and (Minkowski)$_{d-2}\times S^2$ vacua supported by a 2-form $F_{(2)}$. In this section, we shall show that these vacua are all supersymmetric.

Consider first the $(\text{Minkowski})_{d-3}\times S^3$ solution supported by the $H_{(3)}$ field. This is given by

$$
\begin{aligned}
ds_d^2 &= dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{4}{m^2\,(d-1)^2}\, d\Omega_3^2\,,\\
H_{(3)} &= \frac{8}{m^2\,(d-1)^2}\,\Omega_{(3)}\,, \qquad \phi=0\,.
\end{aligned}
\tag{5.1}
$$

If we lift the solution back to D dimensions using the generalised reduction ansatz, it becomes the near-horizon geometry of a $(D-5)$-brane supported by the field $\hat H_{(3)}$. To see this, we start with the $(D-5)$-brane in D dimensions, given by

$$
\begin{aligned}
d\hat s_D^2 &= H^{-\frac{2}{D-2}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{D-4}{D-2}}\,(dr^2 + r^2\, d\Omega_3^2)\,,\\
\hat H_{(3)} &= 2Q\,\Omega_{(3)}\,, \qquad \hat\phi = -\tfrac12 \hat a\, \log H\,, \qquad H = 1 + Q/r^2\,.
\end{aligned}
\tag{5.2}
$$

In the near-horizon limit, the additive constant 1 in H is dropped. Making the coordinate transformation $r^2/Q = e^{(D-2)\,m\,z}$, and letting $Q = 4/((D-2)^2\, m^2)$, we obtain

$$
\begin{aligned}
d\hat s_D^2 &= e^{2mz}\Big(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + dz^2 + \frac{4}{m^2\,(D-2)^2}\, d\Omega_3^2\Big)\,,\\
\hat H_{(3)} &= \frac{8}{m^2\,(D-2)^2}\,\Omega_{(3)}\,, \qquad \hat\phi = \frac{4}{\hat a}\, mz\,,
\end{aligned}
\tag{5.3}
$$

which fits the reduction ansatz precisely, giving rise to the lower-dimensional solution (5.1).

The supersymmetry of the $(\text{Minkowski})_{d-3}\times S^3$ solution is easily established. Firstly, since its lift to $D=d+1$ dimensions gives the near-horizon limit of the $(D-5)$-brane, as discussed above, it is manifest that *qua* D-dimensional solution, it will preserve one half of the D-dimensional supersymmetry. This halving of supersymmetry comes about from the usual projection condition for supersymmetry of the $(D-5)$-brane, $\hat\epsilon = \hat\Gamma_*\,\hat\epsilon$, where $\hat\Gamma_*$ is built from the product of Dirac matrices in the world-volume of the $(D-5)$-brane. As is well known, for any of the BPS brane solutions with metric given by

$$
d\hat s^2 = e^{2A}\, dx^\mu\, dx_\mu + e^{2B}\, dy^m\, dy^m\,,
\tag{5.4}
$$

the Killing spinors are given by

$$
\hat\epsilon = e^{\frac12 A}\,\hat\epsilon_0\,, \qquad \hat\Gamma_*\,\hat\epsilon_0 = \hat\epsilon_0\,,
\tag{5.5}
$$

where $\hat\epsilon_0$ is a constant spinor. We see from (5.3) that $A = mz$, and hence the Killing spinors in D dimensions take the form

$$\hat\epsilon = e^{\frac12 mz}\,\hat\epsilon_0\,. \tag{5.6}$$

Since this z dependence matches precisely the z dependence for $\hat\epsilon$ in the generalised reduction ansatz (4.8), it immediately follows that the $(\text{Minkowski})_{d-3}\times S^3$ solution will be supersymmetric *qua* solution of the d-dimensional gauged supergravity.

Another class of supersymmetric vacuum is of the form $(\text{Minkowski})_{d-2}\times S^2$, supported by one of the two-form field strengths $F^a_{(2)}$. It is given by

$$\begin{aligned} ds_d^2 &= dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{1}{m^2\,(d-1)^2}\, d\Omega_2^2\,,\\ F_{(2)} &= \frac{\sqrt2}{m\,(d-1)}\,\Omega_{(2)}\,,\qquad \phi=0\,. \end{aligned} \tag{5.7}$$

Lifting this solution back to D dimensions, it becomes the near-horizon limit of the $(D-4)$-brane supported by one of the field strengths $\hat F^a_{(2)}$. The $(D-4)$-brane solution is given by

$$\begin{aligned} d\hat s_D^2 &= H^{-\frac{2}{D-2}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{2(D-3)}{D-2}}\,(dr^2 + r^2\, d\Omega_2^2)\,,\\ \hat F_{(2)} &= \sqrt2\, Q\,\Omega_{(2)},\qquad \hat\phi = -\tfrac12 \hat a\,\log H\,,\qquad H = 1+Q/r\,. \end{aligned} \tag{5.8}$$

In the near-horizon limit, the constant 1 in H is dropped. Making the coordinate transformation $r/Q = e^{(D-2)\,mz}$ and setting $Q = 1/(m\,(D-2))$ we have

$$\begin{aligned} d\hat s_D^2 &= e^{2mz}\Big(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + dz^2 + \frac{1}{m^2\,(D-2)^2}\, d\Omega_2^2\Big)\,,\\ \hat F_{(2)} &= \frac{\sqrt2}{m\,(D-2)}\,\Omega_{(2)}\,,\qquad \hat\phi = \frac{4}{\hat a}\, mz\,. \end{aligned} \tag{5.9}$$

This clearly fits the reduction ansatz exactly to give rise to (5.7).

Again, the supersymmetry of the solution as a lifted D-dimensional configuration is manifest, since it is just the near-horizon limit of a BPS $(D-4)$-brane. Its supersymmetry as a solution in the $d = D-1$ dimensional gauged supergravity itself is again easily seen, from the general form (5.5) of the Killing spinors in the lifted $(D-4)$-brane. Thus we again find that the D-dimensional Killing spinors are of the form (5.6), and so comparison with the generalised reduction ansatz (4.8) for $\hat\epsilon$ shows that the $(\text{Minkowski})_{d-2}\times S^2$ solution will be supersymmetric in the d-dimensional gauged supergravity.

6 Supersymmetric time-dependent solutions and pp-waves

In this section we construct a time-dependent solution of the new gauged nine-dimensional supergravity, and we show that it is supersymmetric. It can be thought of as a cosmological solution in the gauged supergravity.

The solution is of a form analogous to a standard domain wall, except that here the "transverse space coordinate" is timelike rather than spatial. It is easily seen that the configuration

$$\begin{aligned} ds_9^2 &= -dt^2 + (\tfrac{8}{7}m\,t)^2 dx^i dx^i\,, \\ e^{\frac{1}{\sqrt{14}}\phi} &= \tfrac{8}{7}m\,t\,. \end{aligned} \tag{6.1}$$

solves the nine-dimensional equations of motion that follow from (2.17). Note that the form-fields are all zero in this solution.

The fermionic transformation rules (4.10) in this background reduce to

$$\begin{aligned} \delta\lambda &= -\tfrac{1}{2\sqrt{2}}\gamma^M(\partial_M\phi)\,\epsilon - \tfrac{4\mathrm{i}}{\sqrt{7}}me^{-\frac{1}{\sqrt{14}}\phi}\epsilon\,, \\ \delta\psi_M &= \nabla_M\,\epsilon + \tfrac{4\mathrm{i}}{7}me^{-\frac{1}{\sqrt{14}}\phi}\gamma_M\,\epsilon\,, \end{aligned} \tag{6.2}$$

and it is easily verified that (6.1) is supersymmetric.

In the string frame, the metric in the solution (6.1) becomes simply the Minkowski metric $ds^2_{\rm str} = \eta_{MN}dx^M dx^N$, where

$$t = \exp(\tfrac{8}{7}m\,x^0)\,. \tag{6.3}$$

The dilaton is a linear function of the redefined time; $\Phi = -4mx^0 +$ constant.

The solution (6.1) is straightforwardly lifted to ten dimensions, where it gives

$$\begin{aligned} ds_{10}^2 &= e^{2mz}\Big[-(\tfrac{8}{7}m\,t)^{-1/4}dt^2 + (\tfrac{8}{7}m\,t)^{7/4}(dz^2 + dx^i dx^i)\Big]\,, \\ e^{\hat\phi} &= e^{4mz}(\tfrac{8}{7}m\,t)^{7/2}\,. \end{aligned} \tag{6.4}$$

This can again be viewed as a time-dependent supersymmetric cosmological solution, driven purely by the dilaton. In the string frame the metric is again Minkowskian, but now the dilaton is linearly proportional to the light-cone coordinate x^+:

$$ds^2_{\rm str} = 2dx^+\,dx^- + dx^i\,dx^i\,, \qquad \Phi = x^+\,. \tag{6.5}$$

A metric-dilaton configuration of this kind was also discussed in [15]. It is straightforward to see that the solution preserves half of the supersymmetry, with the Killing spinor given by $\gamma_+\,\epsilon_0$ where ϵ_0 is a constant spinor.

A further uplift to $D=11$ using the standard Kaluza-Klein formula

$$ds_{11}^2 = e^{\frac{1}{6}\hat\phi} ds_{10}^2 + e^{-\frac{4}{3}\hat\phi} dy^2 \tag{6.6}$$

yields the Ricci-flat solution

$$ds_{11}^2 = -r^2 dt^2 + t^2 dr^2 + r^2 t^2 dx^i dx^i + r^{-4} t^{-4} dy^2\,, \tag{6.7}$$

where we have changed from the ten-dimensional coordinate z to a new coordinate r defined by $r = e^{\frac{4}{3}mz}(\frac{8}{7}m\,t)^{1/6}$. The metric (6.7) is a pp-wave. To see this, we introduce new coordinates X_+ and X_- defined by

$$r^2\,t^2 = X_+\,, \qquad \frac{r}{t} = e^{2X_-}\,, \tag{6.8}$$

in terms of which (6.7) becomes

$$ds_{11}^2 = dX_+ dX_- + X_+ dx^i dx^i + X_+^{-2} dy^2\,. \tag{6.9}$$

Thus, we conclude that in eleven dimensions the solution describes a pp-wave.

The metric (6.9) is a particular example of a more general class of pp-waves, contained within the ansatz

$$ds_D = dX_+ dX_- + X_+^{h_1} dx^{m_1} dx^{m_1} + X_+^{h_2} dy^{m_2} dy^{m_2} + X_+^{h_3} dz^{m_3} dz^{m_3} + \cdots\,. \tag{6.10}$$

Here, we take the index ranges to be

$$1 \le m_1\ \le p_1\,, \qquad p_1 + 1 \le m_2\ \le p_1 + p_2\,, \qquad \text{etc.}\,, \tag{6.11}$$

and so the total dimension is $D = 2 + p_1 + p_2 + \cdots$. The only non-vanishing vielbein components of the Riemann tensor for (6.10) are given by

$$R_{m_i\,+\,m_j\,+} = -\tfrac{1}{2} h_i (h_i - 2) X_+^{-2} \delta_{m_i\,m_j}\,. \tag{6.12}$$

Thus (6.10) is Ricci-flat if

$$0 = \sum_{i=1} p_i h_i (h_i - 2)\,. \tag{6.13}$$

The pp-wave (6.9) that resulted from lifting our time-dependent cosmological solution to $D=11$ is the special case with

$$p_1 = 8\,, \qquad h_1 = 1\,, \qquad p_2 = 1\,, \qquad h_2 = -2\,, \tag{6.14}$$

which clearly satisfies (6.13).

7 Conclusions

In this paper, we have obtained generalised Kaluza-Klein reductions of the low-energy effective actions of string theories involving the metric, the dilaton, a 3-form field strength and a 2-form field strength. The generalised reduction gauges two global symmetries, namely the homogeneous scaling symmetry of the equations of motion, and also the dilaton shift symmetry of the Lagrangian. The resulting dimensionally-reduced theory has a positive scalar potential, in the form of a single-exponential of the lower-dimensional dilaton. We showed that the reduction is supersymmetric, by explicitly deriving the lower-dimensional supersymmetry transformation rules.

Although it might seem somewhat perverse to perform generalised reductions of the kind we have considered in this paper, they are actually related by U-duality to more conventional reductions that have been considered extensively in the past. Specifically, a generalised reduction involving the global shift symmetry of the axion in the type IIB theory has been used in order to establish a T-duality between the type IIB theory and the massive type IIA theory [9]. The S-duality of the type IIB theory implies that one should also consider $SL(2,R)$-related generalised reductions [11], which will involve the global shift symmetry of the dilaton. When one extends the discussion of non-perturbative dualities to lower dimensions, the underlying global Cremmer-Julia type symmetries can only be interpreted as strictly internal symmetries if one also makes use of the scaling symmetry of the equations of motion that homogeneously scales the Lagrangian. Thus it is very natural to consider generalised reductions of the kind we have studied in this paper.

The new supergravities have the interesting feature that they all admit supersymmetric vacuum solutions of the form (Minkowski)$\times S^3$, and in some cases also (Minkowski)$\times S^2$. These solutions provide novel compatifications of higher dimensional string theories. Furthermore, owing to the positivity of the scalar potential, the supergravities we have obtained admit time-dependent cosmological solutions that preserve half of the supersymmetry. Lifting these solutions back to $D = 10$, they yield supersymmetric time-dependent solutions driven purely by the dilaton, with no form-field fluxes. Under a further lifting to eleven dimensions, these time-dependent solutions become supersymmetric pp-waves. It would be interesting to study string

theory and M-theory in these simple but non-trivial backgrounds.

A Bosonic reduction ansatz; Einstein frame

We begin by reducing the $D = d+1$ dimensional Ricci tensor to d dimensions by using the metric ansatz in (2.5). We choose the natural vielbein basis

$$\hat{e}^a = e^{m_2 z + \alpha\varphi} e^a , \qquad \hat{e}^z = e^{m_2 z + \beta\varphi}(dz + \mathcal{A}_{(1)}) . \tag{A.1}$$

Thus we have

$$\hat{e}_M{}^A = e^{m_2 z} \begin{pmatrix} e^{\alpha\varphi} e_\mu{}^a & e^{\beta\varphi}\mathcal{A}_\mu \\ 0 & e^{\beta\varphi} \end{pmatrix} , \qquad \hat{e}_A{}^M = e^{-m_2 z} \begin{pmatrix} e^{-\alpha\varphi} e_a{}^\mu & -e^{-\alpha\varphi}\mathcal{A}_a \\ 0 & e^{-\beta\varphi} \end{pmatrix} . \tag{A.2}$$

The determinant of the metric is

$$\sqrt{-\hat{g}} = e^{(d+1)m_2 z + (\beta + d\alpha)\varphi}\sqrt{-g} = e^{(d+1)m_2 z + 2\alpha\varphi}\sqrt{-g} . \tag{A.3}$$

Using the first Cartan structure equation with zero torsion, $d\hat{e}^A = -\hat{\omega}^A{}_B \wedge \hat{e}^B$, we obtain the spin connections

$$\begin{aligned} \hat{\omega}^a{}_b &= \omega^a{}_b + e^{-(m_2 z + \alpha\varphi)}\Big((\alpha\partial_b\varphi - m_2\mathcal{A}_b)\,\hat{e}^a - (\alpha\partial^a\varphi - m_2\mathcal{A}^a)\,\hat{e}_b\Big) \\ &\quad -\tfrac{1}{2}e^{-m_2 z + (\beta - 2\alpha)\varphi}\mathcal{F}^a{}_b\,\hat{e}^z , \\ \hat{\omega}^a{}_z &= e^{-(m_2 z + \alpha\varphi)}(m_2\mathcal{A}^a - \beta\partial^a\varphi)\,\hat{e}^z - \tfrac{1}{2}e^{-m_2 z + (\beta - 2\alpha)\varphi}\mathcal{F}^a{}_b\,\hat{e}^b + m_2 e^{-(m_2 z + \beta\varphi)}\hat{e}^a . \end{aligned} \tag{A.4}$$

From the curvature 2-forms $\hat{\Theta}^A{}_B = d\hat{\omega}^A{}_B + \hat{\omega}^A{}_C \wedge \hat{\omega}^C{}_B = \frac{1}{2}\hat{R}^A{}_{BCD}\hat{e}^C \wedge \hat{e}^D$, we obtain the Ricci tensor with vielbein components

$$\begin{aligned} \hat{R}_{ab} &= e^{-2(m_2 z + \alpha\varphi)}\Big(R_{ab} - \tfrac{1}{2}\partial_a\varphi\,\partial_b\varphi - \alpha\eta_{ab}\Box\varphi \\ &\quad + \alpha m_2(d-1)(\mathcal{A}^c\partial_c\varphi\,\eta_{ab} - \mathcal{A}_a\partial_b\varphi - \mathcal{A}_b\partial_a\varphi) \\ &\quad + \tfrac{1}{2}m_2(d-1)(\nabla_a\mathcal{A}_b + \nabla_b\mathcal{A}_a) + m_2\nabla_c\mathcal{A}^c\eta_{ab} + m_2^2(d-1)(\mathcal{A}_a\mathcal{A}_b - \mathcal{A}_{(1)}^2\eta_{ab})\Big) \\ &\quad - m_2^2(d-1)e^{-2(m_2 z + \beta\varphi)}\eta_{ab} - \tfrac{1}{2}e^{-2(m_2 z + d\alpha\varphi)}\mathcal{F}_a{}^c\mathcal{F}_{bc} , \\ \hat{R}_{az} &= e^{-2m_2 z + (d-3)\alpha\varphi}\Big(\tfrac{1}{2}\nabla^b(e^{-2(d-1)\alpha\varphi}\mathcal{F}_{ab}) + m_2(d-1)(\beta\partial_a\varphi - m_2\mathcal{A}_a)\Big) \\ &\quad - \tfrac{1}{2}m_2(d-1)e^{-2m_2 z - (d+1)\alpha\varphi}\mathcal{A}^b\mathcal{F}_{ab} , \\ \hat{R}_{zz} &= e^{-2(m_2 z + \alpha\varphi)}\Big(- \beta\Box\varphi + m_2\nabla_c\mathcal{A}^c + m_2\beta(d-1)\mathcal{A}^b\partial_b\varphi - m_2^2(d-1)\mathcal{A}_{(1)}^2\Big) \\ &\quad + \tfrac{1}{4}e^{-2(m_2 z + d\alpha\varphi)}\mathcal{F}_{(2)}^2 . \end{aligned} \tag{A.5}$$

The Ricci scalar is

$$\begin{aligned}\hat{R} &= e^{-2(m_2 z+\alpha\varphi)}\Big(R - 2\alpha\,\Box\varphi - \tfrac{1}{2}(\partial\varphi)^2 + 2m_2 d\,\nabla_a\mathcal{A}^a - m_2^2 d\,(d-1)\mathcal{A}_{(1)}^2\Big)\\ &\quad - e^{-2m_2 z}\Big(m_2^2\,d\,(d-1)e^{-2\beta\varphi} + \tfrac{1}{4}e^{-2d\alpha\varphi}\mathcal{F}_{(2)}^2\Big)\,. \qquad (A.6)\end{aligned}$$

The reduced Ricci components in (A.5) have been simplified through use of the relations (2.6).

The Laplacian operator acting on the D-dimensional dilaton is given by

$$e^{2m_2 z+2\alpha\varphi}\,\widehat{\Box}\hat{\phi} = \Box\phi - m_2(d-1)\Big(\mathcal{A}^\mu\partial_\mu\phi - \frac{4}{\hat{a}}\,m_1\,(\mathcal{A}_{(1)}^2 + e^{2(d-1)\alpha\varphi})\Big) - \frac{4}{\hat{a}}\,m_1\nabla_\mu\mathcal{A}^\mu\,, \quad (A.7)$$

where $\hat{\phi} = \phi + \frac{4}{\hat{a}}\,m_1 z$, as given by (2.5).

The vielbein components of the various D-dimensional antisymmetric tensors reduce according to

$$\begin{aligned}\hat{H}_{a_1\cdots a_n} &= e^{-(m_2+(n-1)m_1)z-n\alpha\varphi}\,H_{a_1\cdots a_n}\,,\\ \hat{H}_{a_1\cdots a_{n-1}z} &= e^{-(m_2+(n-1)m_1)z+(d-n-1)\alpha\varphi}\,H_{a_1\cdots a_{n-1}}\,. \qquad (A.8)\end{aligned}$$

B Fermionic reduction ansatz in $D \le 10$; Einstein frame

In this appendix we provide an arbitrary dimensional generalised ansatz that reduces the fermions in $D = d+1$ to d dimensions. The generalised ansatz we are constructing is such that the standard S^1 reduction ($m_1 = 0 = m_2$) reduces canonical fermionic kinetic terms with a normalization as

$$\hat{e}^{-1}\hat{\mathcal{L}} = \kappa(\hat{\bar{\Psi}}_M\hat{\gamma}^{MNP}\widehat{\nabla}_N\hat{\Psi}_P + \hat{\bar{\lambda}}\hat{\gamma}^M\widehat{\nabla}_M\hat{\lambda}) \qquad (B.1)$$

to canonical kinetic terms

$$e^{-1}\mathcal{L} = \kappa(\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \bar{\chi}\gamma^\mu\nabla_\mu\chi) + \text{rest}\,. \qquad (B.2)$$

Here κ is an arbitrary coefficient. Performing the split of the gravitino as $\hat{\psi}_A = (\hat{\psi}_a, \hat{\psi}_D)$ an ansatz that accomplishes this is

$$\hat{\epsilon} = e^{\frac{1}{2}m_2 z}e^{\frac{1}{2}\alpha\varphi}\,\epsilon\,,$$

$$
\begin{aligned}
\hat{\lambda} &= \tfrac{1}{\sqrt{D-2}}\, e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi}(\chi + \sqrt{D-3}\,\lambda)\,,\\
\hat{\psi}_D &= \tfrac{\sqrt{D-3}}{D-2} e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi}\gamma_D(\sqrt{D-3}\,\chi - \lambda)\,,\\
\hat{\psi}_a &= e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi}(\psi_a - \tfrac{1}{(D-2)\sqrt{D-3}}\gamma_a(\sqrt{D-3}\,\chi - \lambda))\,,\\
\hat{\phi} &= \sqrt{\tfrac{D-3}{D-2}}\,\phi_1 + \tfrac{1}{\sqrt{D-2}}\phi_2 + \sqrt{2(D-2)}\,m_1 z\,,\\
\varphi &= -\tfrac{1}{\sqrt{D-2}}\phi_1 + \sqrt{\tfrac{D-3}{D-2}}\phi_2\,.
\end{aligned}
\tag{B.3}
$$

Note that, here and elsewhere in this paper our convention is always $\alpha > 0$. A consistent truncation of the matter multiplet can be obtained by setting $m_1 = m_2$ and $\phi_2 = 0 = \chi$.

C Einstein-frame to string-frame conversion

The D-dimensional Lagrangian in the Einstein frame is given by

$$
\begin{aligned}
e^{-1}\mathcal{L} &= R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{\hat{a}\phi}H_{(3)}^2 - \tfrac{1}{4}e^{\frac{1}{2}\hat{a}\phi}(F_{(2)}^a)^2 - \tfrac{1}{2}\bar{\Psi}_M\gamma^{MNP}\nabla_N\Psi_P - \tfrac{1}{2}\bar{\lambda}\,\gamma^M\nabla_M\lambda\\
&\quad -\tfrac{1}{2\sqrt{2}}\bar{\lambda}\gamma^N\gamma^M\Psi_N\partial_M\phi + \cdots\,,
\end{aligned}
\tag{C.1}
$$

where $\hat{a} = \sqrt{\frac{8}{D-2}}$, and where we have omitted additional interaction and four-fermi terms. This may be mapped to the string frame Lagrangian

$$
\begin{aligned}
\tilde{e}^{-1}\widetilde{\mathcal{L}} &= e^{-2\Phi}\Big(\widetilde{R} + 4(\partial\Phi)^2 - \tfrac{1}{12}\widetilde{H}_{(3)}^2 - \tfrac{1}{4}(\widetilde{F}_{(2)}^a)^2 - \tfrac{1}{2}\bar{\tilde{\Psi}}_M\tilde{\gamma}^{MNP}\widetilde{\nabla}_N\tilde{\Psi}_P - \tfrac{1}{2}\bar{\tilde{\lambda}}\,\tilde{\gamma}^M\widetilde{\nabla}_M\tilde{\lambda}\\
&\quad -(\bar{\tilde{\Psi}}_N\tilde{\gamma}^N\tilde{\Psi}^M - \frac{\hat{a}}{2\sqrt{2}}\bar{\tilde{\lambda}}\tilde{\gamma}^N\tilde{\gamma}^M\tilde{\Psi}_N)\partial_M\Phi + \cdots\Big)\,,
\end{aligned}
\tag{C.2}
$$

by the transformations

$$
\begin{aligned}
g_{MN} &= e^{\frac{1}{2}\hat{a}\phi}\,\tilde{g}_{MN}\,, \qquad H_{MNP} = \widetilde{H}_{MNP}\,, \qquad F_{MN}^a = \widetilde{F}_{MN}^a\,, \qquad \phi = -\hat{a}\,\Phi\,,\\
\epsilon &= e^{\frac{1}{8}\hat{a}\phi}\,\tilde{\epsilon}\,, \qquad \lambda = e^{-\frac{1}{8}\hat{a}\phi}\tilde{\lambda}\,, \qquad \Psi_M = e^{\frac{1}{8}\hat{a}\phi}\tilde{\Psi}_M\,.
\end{aligned}
\tag{C.3}
$$

Note that $\gamma_M = e^{\frac{1}{4}\hat{a}\phi}\,\tilde{\gamma}_M$ i.e. $\gamma_A = \tilde{\gamma}_A$. Furthermore, we have made use of the D-dimensional Majorana flip properties $\bar{\psi}\gamma^M\chi = -\bar{\chi}\gamma^M\psi$ and $\bar{\psi}\gamma^{MNP}\chi = \bar{\chi}\gamma^{MNP}\psi$ for any two anti-commuting spinors ψ and χ.

The bosonic reduction ansätze in the string frame are considerably simpler than their Einstein-frame counterparts. The reduction of the $D = d + 1$ dimensional Ricci

tensor is given by

$$\begin{aligned}
\hat{R}_{ab} &= R_{ab} + \tfrac{1}{\sqrt{2}}\nabla_a\partial_b\varphi - \tfrac{1}{2}\partial_a\varphi\partial_b\varphi - \tfrac{1}{2}e^{-\sqrt{2}\varphi}\mathcal{F}_{ac}\,\mathcal{F}_b{}^c\,,\\
\hat{R}_{az} &= \tfrac{1}{2}e^{\sqrt{2}\varphi}\,\nabla^b(e^{-\frac{3}{\sqrt{2}}\varphi}\mathcal{F}_{ab})\,,\\
\hat{R}_{zz} &= \tfrac{1}{\sqrt{2}}\Box\varphi - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{4}e^{-\sqrt{2}\varphi}\mathcal{F}^2_{(2)}\,,\\
\hat{R} &= R + \sqrt{2}\Box\varphi - (\partial\varphi)^2 - \tfrac{1}{4}e^{-\sqrt{2}\varphi}\mathcal{F}^2_{(2)}\,.
\end{aligned} \tag{C.4}$$

Some useful formulae for the reduction of the scalar fields are:

$$\begin{aligned}
\hat{\Box}\hat{\Phi} = \hat{\Box}\Big(\Phi - \frac{\varphi}{\sqrt{8}} - \tfrac{1}{2}(D-2)mz\Big) &= \Box\Phi - \tfrac{1}{\sqrt{8}}\Box\varphi - \tfrac{1}{\sqrt{2}}(\partial_\mu\varphi\,\partial^\mu\Phi - \tfrac{1}{\sqrt{8}}(\partial\varphi)^2)\\
&\quad - \tfrac{1}{2}m(d-1)(\tfrac{1}{\sqrt{2}}\,\mathcal{A}^\mu\partial_\mu\varphi - \nabla_\mu\mathcal{A}^\mu)\,,
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
(\partial\hat{\Phi})^2 &= (\partial\Phi)^2 + \tfrac{1}{8}(\partial\varphi)^2 - \tfrac{1}{\sqrt{2}}\,\partial_\mu\varphi\,\partial^\mu\Phi + m(d-1)\mathcal{A}^\mu(\partial_\mu\Phi - \tfrac{1}{\sqrt{8}}\partial_\mu\varphi)\\
&\quad + \tfrac{1}{4}m^2(d-1)^2(\mathcal{A}^2_{(1)} + e^{\sqrt{2}\varphi})\,,
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
\hat{e}_a{}^M\hat{e}_b{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} &= \nabla_a\partial_b\Phi - \tfrac{1}{\sqrt{8}}\nabla_a\partial_b\varphi + \tfrac{1}{4}m(d-1)(\nabla_a\mathcal{A}_b + \nabla_b\mathcal{A}_a)\,,\\
\hat{e}_a{}^M\hat{e}_z{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} &= -\tfrac{1}{2}e^{-\frac{1}{\sqrt{2}}\varphi}\mathcal{F}_a{}^b(\partial_b\Phi - \tfrac{1}{\sqrt{8}}\,\partial_b\varphi) - \tfrac{1}{2\sqrt{2}}m(d-1)e^{\frac{1}{\sqrt{2}}\varphi}\partial_a\varphi\\
&\quad - \tfrac{1}{4}m(d-1)e^{-\frac{1}{\sqrt{2}}\varphi}\mathcal{F}_{ab}\mathcal{A}^b\,,\\
\hat{e}_z{}^M\hat{e}_z{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} &= -\tfrac{1}{\sqrt{2}}\partial^\mu\varphi\,(\partial_\mu\Phi - \tfrac{1}{\sqrt{8}}\,\partial_\mu\varphi) - \tfrac{1}{2\sqrt{2}}m(d-1)\mathcal{A}^\mu\partial_\mu\varphi\,.
\end{aligned} \tag{C.7}$$

References

[1] A. Salam and E. Sezgin, *Chiral compactification on Minkowski*$\times S^2$ *Of* $N=2$ *Einstein-Maxwell supergravity in six dimensions,* Phys. Lett. **B147**, 47 (1984).

[2] Y. Aghababaie, C.P. Burgess, S.L. Parameswaran and F. Quevedo, *SUSY breaking and moduli stabilization from fluxes in gauged* $6D$ *supergravity,* JHEP **0303**, 032 (2003), hep-th/0212091.

[3] Y. Aghababaie, C.P. Burgess, S.L. Parameswaran and F. Quevedo, *Towards a naturally small cosmological constant from branes in* $6D$ *supergravity,* Nucl. Phys. **B680**, 389 (2004), hep-th/0304256.

[4] M. Cvetič, G.W. Gibbons and C.N. Pope, *A string and M-theory origin for the Salam-Sezgin model,* Nucl. Phys. **B677**, 164 (2004), hep-th/0308026.

[5] G.W. Gibbons and C.N. Pope, *Consistent S^2 Pauli reduction of six-dimensional chiral gauged Einstein-Maxwell supergravity,* hep-th/0307052.

[6] I.V. Lavrinenko, H. Lü and C.N. Pope, *Fibre bundles and generalised dimensional reductions,* Class. Quant. Grav. **15**, 2239 (1998), hep-th/9710243.

[7] J. Kerimo and H. Lü, *New $D = 6$, $\mathcal{N} = (1,1)$ gauged supergravity with supersymmetric (Minkowski)$_4 \times S^2$ vacuum*, Phys. Lett. **B576**, 219 (2003), hep-th/0307222.

[8] J. Kerimo, J.T. Liu, H. Lü and C.N. Pope, *Variant $\mathcal{N} = (1,1)$ Supergravity and (Minkowski)$_4 \times S^2$ Vacua*, hep-th/0401001.

[9] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, *Duality of Type II 7-branes and 8-branes,* Nucl. Phys. **B470**, 113 (1996), hep-th/9601150.

[10] P.M. Cowdall, H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, *Domain walls in massive supergravities,* Nucl. Phys. **B486**, 49 (1997), hep-th/9608173.

[11] P. Meessen and T. Ortin, *An $Sl(2,Z)$ multiplet of nine-dimensional type II supergravity theories*, Nucl. Phys. **B541**, 195 (1999), hep-th/9806120.

[12] E. Bergshoeff, T. de Wit, U. Gran, R. Linares and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions,* JHEP **0210**, 061 (2002), hep-th/ 0209205.

[13] P.S. Howe, N.D. Lambert and P.C. West, *A new massive type IIA supergravity from compactification,* Phys. Lett. **B416**, 303 (1998), hep-th/9707139.

[14] L.J. Romans, *Massive $N = 2a$ supergravity in ten-dimensions,* Phys. Lett. **B169**, 374 (1986).

[15] G. Papadopoulos, J.G. Russo and A.A. Tseytlin, *Solvable model of strings in a time-dependent plane-wave background,* Class. Quant. Grav. **20**, 969 (2003), hep-th/0211289.